Extract Shape Characteristic Points From Cubic B-spline Curve By Segmented Cubic Bézier Curve

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***Abstract*—After analyzing the curvature expression, the inflection points were given by the known planar cubic Bézier control polygon information. Based on that, we proposed a brief algorithm that can obtain the shape feature points of cubic Bézier and B-spline curves. Experimental results show that the method is rapid, accurate, and robust.**

I. Introduction

In practice application, it is necessary to accurately describe the shape characteristic of the parametric curve. In general, It mostly depends on the analysis of the Bernstein basis function and control polygon to describe the shape characteristic of Bézier curve [1]. When peoples need to divide some segmented convex curve from the shape characteristic points [2], the method of extracting the characteristic points based on curvature expression have to be considered.

As forthgoers, M.Sakai [3], FalaiChen[4], Qinming Yang[5] proposed some better algorithms to get inflection points of parametric curve, but they don't discuss the constraints from control polygon on the inflection point, which leads to the implementation of algorithm of inflection point is more complex.

In order to quickly and accurately obtain the shape characteristic points of planar cubic Bézier curve and cubic B-spline curve, according to the nu-uniform inflection points concept in some branch of mathematics [3][4][5], we classified inflection points into singular points and non-singular inflection points [6]. Then, some specific formula for calculating inflection points were given and an effective algorithm was proposed.

II. Shape Feature of Cubic Bézier

Curvature is an important parameter of planar curve. In this section, in order to quickly and accurately obtain the shape characteristic points, formula for calculating inflection points that can be classified into singular points and non-singular inflection points would be derived by analyzing curvature expression and control polygon of cubic Bézier curve [7].

A. Curvature of Cubic Bézier Curve

The cubic Bézier curve is defined as follow:

$$\boldsymbol{C}(t) = \sum_{i=0}^{3} B_i^3(t)\boldsymbol{P_i} \qquad t \in [0,1] \tag{1}$$

Here, $\boldsymbol{P_i}$ are control points of Bézier curve, they constitute control polygon of Bézier curve. $B_3^i(t)$ are cubic Bernstein basis function shown as follow:

$$B_i^3(t) = \frac{3!}{i!(3-i)!} t^i (1-t)^{3-i} \qquad i = (0,\dots,3) \tag{2}$$

When the curve lies on the XY plane, curvature can be expressed by equation (3):

$$\kappa(t) = \frac{|\boldsymbol{C}\,'(t) \times \boldsymbol{C}\,''(t)|}{(|\boldsymbol{C}\,'(t)|)^3} \tag{3}$$

Here, $\boldsymbol{C}\,'(t)$ is first derivative of $\boldsymbol{C}(t)$. It shows the velocity in physics and shows tangential direction in the geometry on the point $\boldsymbol{C}(t)$. The first derivative of cubic Bézier curve can be expressed by equation (4):

$$\begin{aligned} \boldsymbol{C}\,'(t) &= 3\sum_{i=0}^{2} B_i^2(t)(\boldsymbol{P_{i+1}} - \boldsymbol{P_i}) \\ &= 3(\boldsymbol{A_0} + 2t\boldsymbol{D_0} + t^2\boldsymbol{E_0}) \end{aligned} \tag{4}$$

$\boldsymbol{C}\,''(t)$ is the second derivative of $\boldsymbol{C}(t)$. It shows the acceleration in physics and shows curved tendency in the geometry. The second derivative of cubic Bézier curve can be expressed by equation (5):

$$\begin{aligned} \boldsymbol{C}\,''(t) &= 6((1-t)\boldsymbol{D_0} + t\boldsymbol{D_1}) \\ &= 6(\boldsymbol{D_0} + t\boldsymbol{E_0}) \end{aligned} \tag{5}$$

In equation (4) and (5):

$$\begin{aligned} \boldsymbol{A_i} &= \boldsymbol{P_{i+1}} - \boldsymbol{P_i} \qquad (\boldsymbol{i} = 0,1,2) \\ \boldsymbol{D_j} &= \boldsymbol{A_{j+1}} - \boldsymbol{A_j} \qquad (\boldsymbol{j} = 0,1) \\ \boldsymbol{E_k} &= \boldsymbol{D_{k+1}} - \boldsymbol{D_k} \qquad (\boldsymbol{k} = 0) \end{aligned} \tag{6}$$

Here, $\boldsymbol{A_i}$, $\boldsymbol{D_j}$ and $\boldsymbol{E_k}$ are called first-order, second-order and third-order control vector of cubic Bézier curve, individually. Fig.1 shows the case that $\boldsymbol{P_0}$ and $\boldsymbol{P_3}$ lying on the same side of $\boldsymbol{A_1}$.

Form the equation (4) and (5), $\boldsymbol{C}\,'(t)$ and $\boldsymbol{C}\,''(t)$ are continuous and derivable in the parametric interval $t \in [0,1]$. So $\boldsymbol{C}\,'(t) \times \boldsymbol{C}\,''(t)$ shown in equation (3) is also continuous and derivable in the defined interval.

According to the theory of CAGD, $\boldsymbol{C}(t)$ are ***inflection points*** of curve when $\boldsymbol{C}\,'(t) \times \boldsymbol{C}\,''(t) = 0$. For vectors $\boldsymbol{C}\,'(t)$ and

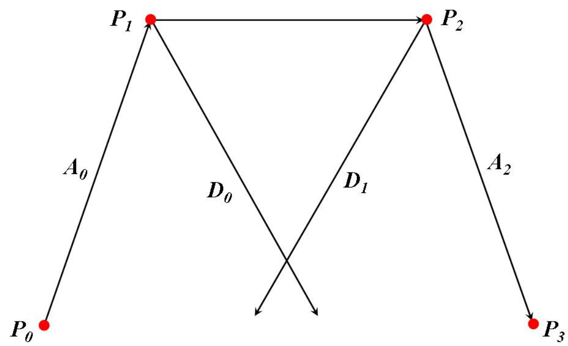

Fig. 1. The geometric meaning of Control point vector

$\boldsymbol{C}''(t)$, there are only two possibility can lead to their cross product is 0:

- $\boldsymbol{C}'(t) \times \boldsymbol{C}''(t) = 0$ and $\boldsymbol{C}'(t) \neq 0$: then $\kappa(t) = 0$. Vectors $\boldsymbol{C}'(t_-) \times \boldsymbol{C}''(t_-)$ and $\boldsymbol{C}'(t_+) \times \boldsymbol{C}''(t_+)$ should have the opposite direction for $\boldsymbol{C}'(t)$. So $\boldsymbol{C}(t)$ are called ***non-singular inflection points***.
- $\boldsymbol{C}'(t) \times \boldsymbol{C}''(t) = 0$ and $\boldsymbol{C}'(t) = 0$: then $\kappa(t) = \frac{0}{0}$, which leads to the curvature of the point can not be calculated. According to the continuity of $\boldsymbol{C}'(t)$, $\boldsymbol{C}'(t_-)$ and $\boldsymbol{C}'(t_+)$ should have the opposite direction for cubic Bézier curve. In general, the points are called the ***singular points*** if they satisfied $\boldsymbol{C}'(t) = 0$. So singular points are also special inflection points.

In this paper, inflection points are called shape characteristic points of curve that contain the non-singular inflection points and singular points.

When use $\boldsymbol{A_{0x}}, \boldsymbol{A_{0y}}, \boldsymbol{D_{0x}}, \boldsymbol{D_{0y}}, \boldsymbol{E_{0x}}, \boldsymbol{E_{0y}}$ represent respectively the x and y components of vectors $\boldsymbol{A_0}$, $\boldsymbol{D_0}$ and $\boldsymbol{E_0}$, if:

$$\begin{aligned} o &= \boldsymbol{A_{0x}D_{0y}} - \boldsymbol{A_{0y}D_{0x}} \\ p &= \boldsymbol{D_{0x}E_{0y}} - \boldsymbol{D_{0y}E_{0x}} \\ q &= \boldsymbol{A_{0x}E_{0y}} - \boldsymbol{A_{0y}E_{0x}} \end{aligned} \tag{7}$$

Combined equations (4), (5) and (7), necessary and sufficient conditions of inflection points exist can be obtain shown as equation (8):

$$pt^2 + qt + o = 0 \quad (t \in [0,1]) \tag{8}$$

The parameter $t \in [0,1]$ can be computed by equation(9):

$$t = \frac{-q \pm \sqrt{q^2 - 4po}}{2p} \quad (t \in [0,1]) \tag{9}$$

Especially, When $\boldsymbol{C}'(t) = 0$, necessary and sufficient conditions of singular points exist can be obtain shown as equation (10):

$$\begin{aligned} t &= \frac{-\boldsymbol{D_{0x}} \pm \sqrt{\boldsymbol{D_{0x}^2} - \boldsymbol{A_{0x}E_{0x}}}}{\boldsymbol{E_{0x}}} \\ &= \frac{-\boldsymbol{D_{0y}} \pm \sqrt{\boldsymbol{D_{0y}^2} - \boldsymbol{A_{0y}E_{0y}}}}{\boldsymbol{E_{0y}}} \end{aligned} \tag{10}$$

B. Control Polygon of Cubic Bézier Curve

According the relative position relationship of the four control points for cubic Bézier curve, the control polygon will be one of the following three forms:

1) $\boldsymbol{P_i}$ lying on the same line.
2) $\boldsymbol{P_0}$,$\boldsymbol{P_3}$ lying on both sides of $\boldsymbol{A_1}$.
3) $\mathbf{P_0P_3}$ lying on the same side of $\boldsymbol{A_1}$.

In order to find the shape characteristic points of the curve, all possibilities of the combination of curvature expression and control polygon should be discussed in detail.

Form 1: In the interval $t \in [0,1]$, $\boldsymbol{C}'(t)$ and $\boldsymbol{C}''(t)$ both have two possibilities of zero and non-zero.

- When $\boldsymbol{C}'(t) \neq 0$: then $\boldsymbol{C}''(t) = 0$ or $\boldsymbol{C}''(t) \neq 0$. Because $\boldsymbol{C}''(t)$ and $\boldsymbol{C}'(t)$ lying on the same line, $|\boldsymbol{C}'(t) \times \boldsymbol{C}''(t)| = 0$ is always tenable, that is $\kappa(t) = 0$ and each point on the curve is non-singular inflection point.
- When $\boldsymbol{C}'(t) = 0$, the singular point exist. Its parameter t can be obtained by equation (8).

Form 2: In the interval $t \in [0,1]$, if $\boldsymbol{C}'(t) \neq 0$ is always tenable, then singular points are not exist (***Theorem 1***). If the point which make $\boldsymbol{C}'(t) \times \boldsymbol{C}''(t) = 0$ exist (***Theorem 2***), it must be non-singular inflection point of the curve. Now, we prove the two theorems.

Theorem 1: If $\boldsymbol{P_0}$, $\boldsymbol{P_3}$ lying on both sides of $\boldsymbol{A_1}$, $\boldsymbol{C}'(t) \neq 0$ is always tenable.

Prove: From equation (4), we can obtain equation (11):

$$\boldsymbol{C}'(t) = 3\left(\sum_{i=0}^{2} B_2^i(t)\boldsymbol{P_{i+1}} - \sum_{i=0}^{2} B_2^i(t)\boldsymbol{P_i}\right) \tag{11}$$

which means $\sum_{i=0}^{2} B_2^i(t)\boldsymbol{P_{i+1}}$ and $\sum_{i=0}^{2} B_2^i(t)\boldsymbol{P_i}$ are two quadratic Bézier curve defined by $\boldsymbol{P_1}$, $\boldsymbol{P_2}$, $\boldsymbol{P_3}$ and $\boldsymbol{P_0}$, $\boldsymbol{P_1}$, $\boldsymbol{P_2}$. According the convex hull property, $\sum_{i=0}^{2} B_2^i(t)\boldsymbol{P_{i+1}}$ and $\sum_{i=0}^{2} B_2^i(t)\boldsymbol{P_i}$ must be lying on both sides of $\boldsymbol{A_1}$ (see Fig.2). From equation (11):

- When $t = 0$, $\boldsymbol{C}'(0) = 3\boldsymbol{A_0} \neq 0$.
- When $t = 1$, $\boldsymbol{C}'(1) = 3\boldsymbol{A_2} \neq 0$.
- For the arbitrary parameter $t \in (0,1)$, the start point and the end point of the vector $\boldsymbol{C}'(t)$ must be lying on both sides of $\boldsymbol{A_1}$, so $\boldsymbol{C}'(t) \neq 0$ is tenable.

Theorem 1 is proved.

Theorem 2: If $\boldsymbol{P_0}$, $\boldsymbol{P_3}$ lying on both sides of $\boldsymbol{A_1}$, the point which make $\boldsymbol{C}'(t) \times \boldsymbol{C}''(t) = 0$ $(t \in [0,1])$ must exist.

Prove: From equation (4), (5) and (6):

- When $t = 0$, $\boldsymbol{C}'(0) \times \boldsymbol{C}''(0) = \boldsymbol{A_0} \times \boldsymbol{A_1}$.
- When $t = 1$, $\boldsymbol{C}'(1) \times \boldsymbol{C}''(1) = \boldsymbol{A_1} \times \boldsymbol{A_2}$.

When $\boldsymbol{P_0}$, $\boldsymbol{P_3}$ lying on both sides of $\boldsymbol{A_1}$, $\boldsymbol{A_0} \times \boldsymbol{A_1}$ and $\boldsymbol{A_1} \times \boldsymbol{A_2}$ defined two vectors with opposite direction. According to the Mean Value Theorem, when $\boldsymbol{C}'(t) \times \boldsymbol{C}''(t)$ is continuous and derivable, the point which make $\boldsymbol{C}'(t) \times \boldsymbol{C}''(t) = 0$ $(t \in [0,1])$ must exist. ***Theorem 2*** is proved.

Form 3: In the interval $t \in [0,1]$, $\boldsymbol{D_0}$, $\boldsymbol{D_1}$ always point to the same side of $\boldsymbol{A_1}$, and $\boldsymbol{C}''(t)$ also points to the same side

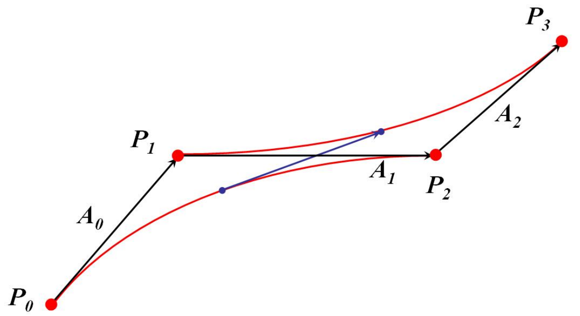

Fig. 2. when $\boldsymbol{P_0}$,$\boldsymbol{P_3}$ lying on the side of $\boldsymbol{A_1}$,geometric meaning of $\boldsymbol{C}\ '(t)$

of $\boldsymbol{A_1}$. So $\boldsymbol{C}\ ''(t) \neq 0$ is always tenable. However, the case $\boldsymbol{C}\ '(t) = 0$(singular points) and $\boldsymbol{C}\ '(t)//\boldsymbol{C}\ ''(t)$(non-singular inflection points) may exist.

For the above three forms of cubic Bézier curve, Form 1 are often not taken into account. There must be at least one inflection point in Form 2. Singular points and non-singular inflection points might exist in Form 3. For the shape characteristic points of cubic Bézier curve, we can use the uniformity algorithm described in the next section to obtain them rapidly.

C. Algorithm for shape characteristic points of Bézier curve

When inflection points (including singular points and non-singular inflection points) were regarded as the shape characteristic points of cubic Bézier curve, according to above discussed results, rapid extraction algorithm of shape characteristic points can be obtained as follows:

Step 1 : According the shape of the control polygon to judge whether it belongs to the Form 1. If it's Form 1, turn to Step 3, otherwise, turn to Step 2.

Step 2 : To implement Step 3 after calculating the parameters and coordinates of inflection points based on equation (8) and (9).

Step 3 : To calculate the parameters and coordinates of inflection points based on equation (10), classified the inflection points into singular points and non-singular inflection points, save their informations.

III. The Shape Characteristic Points of Cubic B-spline curve

Any non-rational cubic B-spline curve can be divided into several segments of the non-rational cubic Bézier curves. So extraction algorithm of shape characteristic points of non-rational cubic B-spline curve can be obtained by dividing it into some cubic Bézier curves.

A. The Segmentation of Cubic B-spline curve

Similar to the Bézier curve, cubic B-spline curve is defined as follows:

$$\boldsymbol{C}(t) = \sum_{i=0}^{n} N_i^3(t)\boldsymbol{P_i} \qquad t \in [0,1] \tag{12}$$

where, $\boldsymbol{P_i}$ are $n+1$ control points of B-spline curve, they constitute control polygon of B-spline curve. If the knot vector is $[x_0, \ldots, x_{n+k+1}]$, the k power B-spline basis function$N_i^k(t)$ is defined as the equation (13)

$$\begin{cases} N_i^0(t) = 1 \qquad (x_i \le t < x_{i+1}) \\ N_i^k(t) = \dfrac{t - x_i}{x_{i+k-1} - x_i} N_i^{k-1} + \dfrac{x_{i+k} - t}{x_{i+k} - x_{i+1}} N_{i+1}^{k-1} \end{cases} \tag{13}$$

For a definited parameter t, when $x_m \le t < x_{m+1}$, in the cubic B-spline basis functions,only the value of $N_{m-3}^3, N_{m-2}^3, N_{m-1}^3, N_m^3$ is non-zero,and the remaining entries are zero. If:

$$\begin{aligned} \alpha &= x_{m+1} - t \\ \beta &= t - x_m \\ \gamma &= x_{m+2} - t \\ \delta &= t - x_{m-1} \\ \epsilon &= x_{m+3} - t \\ \eta &= t - x_{m-2} \\ \lambda &= x_{m+1} - x_m \\ \xi &= x_{m+1} - x_{m-1} \\ \sigma &= x_{m+1} - x_{m-2} \\ \varphi &= x_{m+2} - x_m \\ \psi &= x_{m+2} - x_{m-1} \\ \omega &= x_{m+3} - x_m \\ \theta &= x_{m+3} - x_{m+1} \\ \vartheta &= x_{m+2} - x_{m+1} \\ \nu &= x_m - x_{m-1} \\ \varpi &= x_m - x_{m-2} \end{aligned} \tag{14}$$

so the $N_{m-3}^3, N_{m-2}^3, N_{m-1}^3, N_m^3$ can be changed into the equation (15)

$$\begin{aligned} N_{m-3}^3 &= \frac{\alpha^3}{\sigma\xi\lambda} \\ N_{m-2}^3 &= \frac{\eta\alpha^2}{\sigma\xi\lambda} + \frac{\delta\alpha\gamma}{\psi\xi\lambda} + \frac{\gamma^2\beta}{\psi\varphi\lambda} \\ N_{m-1}^3 &= \frac{\delta^2\alpha}{\psi\xi\lambda} + \frac{\delta\gamma\beta}{\psi\varphi\lambda} + \frac{\epsilon\beta^2}{\omega\varphi\lambda} \\ N_m^3 &= \frac{\beta^3}{\omega\varphi\lambda} \end{aligned} \tag{15}$$

that is means: when $x_m \le t < x_{m+1}$, a cubic B-spline curve can be simplified as following form:

$$\boldsymbol{C}(t) = \sum_{i=m-3}^{m} N_i^3(t)\boldsymbol{P_i} \qquad t \in [x_m, x_{m+1}) \tag{16}$$

If $\mu = \dfrac{\beta}{\lambda}(\mu \in [0,1])$, $(1-\mu) = \dfrac{\alpha}{\lambda}$, the cubic B-spline curve can be described into cubic Bézier curve:

$$\sum_{i=m-3}^{m} N_i^3(t)\boldsymbol{P_i} = \sum_{j=0}^{3} B_j^3(\mu)\boldsymbol{Q_j} \tag{17}$$

By solving the equation(17), we can get:

$$
\begin{aligned}
\boldsymbol{Q_0} &= \frac{\lambda^2}{\sigma\xi}\boldsymbol{P_{m\text{-}3}} + \left(\frac{\nu\varphi}{\psi\xi} + \frac{\varpi\lambda}{\sigma\xi}\right)\boldsymbol{P_{m\text{-}2}} + \frac{\nu^2}{\psi\xi}\boldsymbol{P_{m\text{-}1}} \\
\boldsymbol{Q_1} &= \left(\frac{\varphi}{3\psi} + \frac{2\lambda}{3\xi} + \frac{2\vartheta\nu}{3\xi\psi}\right)\boldsymbol{P_{m\text{-}2}} + \frac{\nu}{\psi}\boldsymbol{P_{m\text{-}1}} \\
\boldsymbol{Q_2} &= \left(\frac{\xi}{3\psi} + \frac{2\lambda}{3\varphi} + \frac{2\vartheta\nu}{3\varphi\psi}\right)\boldsymbol{P_{m\text{-}1}} + \frac{\vartheta}{\psi}\boldsymbol{P_{m\text{-}2}} \\
\boldsymbol{Q_3} &= \frac{\vartheta^2}{\varphi\psi}\boldsymbol{P_{m\text{-}2}} + \left(\frac{\xi\vartheta}{\varphi\psi} + \frac{\lambda\theta}{\varphi\omega}\right)\boldsymbol{P_{m\text{-}1}} + \frac{\lambda^2}{\varphi\omega}\boldsymbol{P_m}
\end{aligned}
\tag{18}
$$

From the above derivation, there are $n+1$ control points of non-rational cubic B-spline curve can be divided into $n-2$ sections of non-rational cubic Bézier curves. Control points of each section of cubic Bézier curve can be obtained by calculating its corresponding four control points of cubic B-spline and their associated knot vector. So the problem about shape characteristic points of cubic B-spline curve can be reduced to shape characteristic points of some cubic Bézier curves.

B. Algorithm of shape characteristic points of B-spline curve

The shape of cubic B-spline curve always be influenced by knot vectors, which is more complex than cubic Bézier curve. When continuous $d(d \geq 2)$ knot vectors have the same values, the two sections of Bézier curve which are divided at piont $\boldsymbol{C}\ (t)$ might exist $\boldsymbol{C}\ '(t)$ is not continuous. Combined with the concept of G-continuation, if $\boldsymbol{C}\ '(t_-)$ and $\boldsymbol{C}\ '(t_+)$ are not in a straight line, curve at this point is G^0 continuation. $\boldsymbol{C}\ (t)$ are called ***cusp point***. If considering cusp points, singular points and non-singular inflection points as the shape characteristic points of cubic B-spline curve, extraction algorithm of shape characteristic points of cubic B-spline curve can be obtained by modified slightly algorithm of cubic Bézier curve:

step 1 : Divide the cubic B-spline curve into some B ’ezier curves according to the equation (18).

step 2 : For each cubic Bézier curve, implement its extraction algorithm of shape characteristic points.

step 3 : Calculating $\kappa_j(1)$,$\kappa_{j+1}(0)$ and $\boldsymbol{C}_j\ '(1)$, $\boldsymbol{C}_{j+1}\ '(0)$ of any two adjacent cubic Bézier curves. If $\boldsymbol{C}_j\ '(1) = 0$ and $\boldsymbol{C}_{j+1}\ '(0) = 0$, the point is a singular point; Otherwise if $\kappa_j(1)$ and $\kappa_{j+1}(0)$ have opposite sign and non-zero, the point is a inflection point; Otherwise if $\boldsymbol{C}_j\ '(1)$ and $\boldsymbol{C}_{j+1}\ '(0)$ are not in a straight line,the point is a cusp point.

step 4 : Sort and merge all shape characteristic points of cubic Bézier curves, the set is shape characteristic points of the cubic B-spline curve.

IV. Experimental Results

From Fig.3 to Fig.5 shows some kinds of shapes characteristic points of cubic Bézier curve included the Form 2 and Form 3. Where the small solid rectangle represents non-singular inflection points and the large hollow rectangular represents singular points.

Fig.3 shown a case that $\boldsymbol{P_0}$, $\boldsymbol{P_3}$ lying both sides of $\boldsymbol{A_1}$. In this case, there must be at least one inflection point. Fig.4 shown

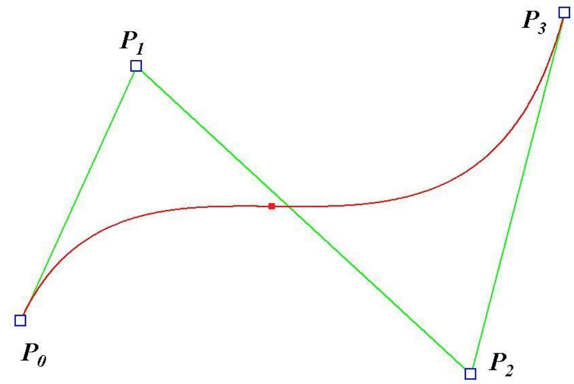

Fig. 3. $\boldsymbol{P_0P_3}$ lying both sides of $\boldsymbol{A_1}$

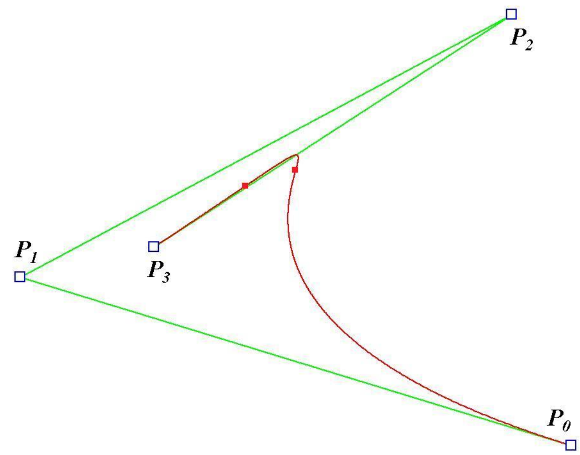

Fig. 4. $\boldsymbol{P_0P_3}$ lying the same side of $\boldsymbol{A_1}$ and $\boldsymbol{A_0}$ did not cross with $\boldsymbol{A_2}$

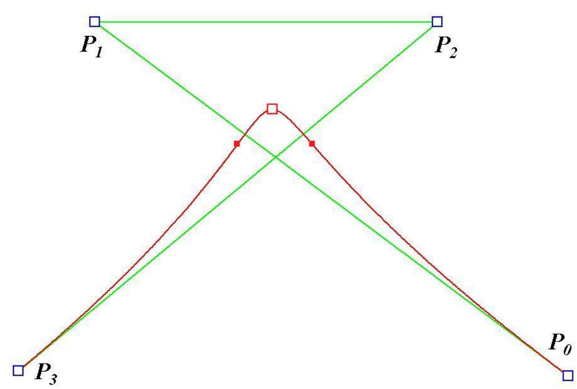

Fig. 5. $\boldsymbol{P_0P_3}$ lying the same side of $\boldsymbol{A_1}$ and $\boldsymbol{A_0}$ crossed with $\boldsymbol{A_2}$

a case that $\boldsymbol{P_0}$, $\boldsymbol{P_3}$ lying on the same side of $\boldsymbol{A_1}$ and $\boldsymbol{A_0}$ did not cross with $\boldsymbol{A_2}$. In this cases, there are two non-singular inflection points. Fig.5 shown a case that $\boldsymbol{P_0}$, $\boldsymbol{P_3}$ lying on the same side of $\boldsymbol{A_1}$ and $\boldsymbol{A_0}$ crossed with $\boldsymbol{A_2}$. In this case, there are two non-singular inflection points and one singular point.

From Fig.6 to Fig.9 shows some kinds of shapes characteristic points of cubic B-spline curve. Where light solid dots represents control points of cubic B-spline curve, light lines connected constitute control polygon of B-spline curve. Dark

TABLE I

SHOWS KNOT VECTORS AND THE NUMBER OF SECTIONS OF CUBIC BÉZIER CURVES FROM FIG.6 TO FIG.9

Basis information	Knot Vector	The number of Bézier curve
Fig.6	{-0.32, -0.25, -0.13, 0.0, 0.2, 0.5, 0.5, 0.8, 1.0, 1.1, 1.22, 1.37}	4
Fig.7	{-0.42, -0.35, -0.23, 0.0, 0.2, 0.5, 0.5, 0.8, 1.0, 1.1, 1.8, 1.87}	4
Fig.8	{0.0, 0.0, 0.0, 0.0, 0.333333, 0.666666, 1.0, 1.0, 1.0, 1.0}	3
Fig.9	{-0.42, -0.35, -0.23, 0.0, 0.5, 0.5, 0.5, 1.0, 1.8, 1.87, 1.95}	2

TABLE II

SHOWS DETAILED INFORMATION FROM FIG.6 TO FIG.9

detailed information	Fig3	Fig4	Fig5	Fig6
$\boldsymbol{P_0}$ The coordinate values	(26, 695)	(31, 322)	(412, 39)	(798, 284)
$\boldsymbol{P_1}$ The coordinate values	(280, 15)	(88, 48)	(113, 19)	(31, 28)
$\boldsymbol{P_2}$ The coordinate values	(558, 645)	(359, 48)	(462, 19)	(345, 28)
$\boldsymbol{P_3}$ The coordinate values	(716, 69)	(80, 663)	(23, 694)	(25, 688)
$\boldsymbol{P_4}$ The coordinate values	(878, 510)	(1126, 663)	(1262, 32)	(1318, 436)
$\boldsymbol{P_5}$ The coordinate values	(982, 91)	(953, 56)	(1262, 636)	(899, 650)
$\boldsymbol{P_6}$ The coordinate values	(1168, 424)	(1200, 56)		(899, 29)
$\boldsymbol{P_7}$ The coordinate values	(1260, 307)	(1253, 326)		
The parameters value of characteristic points	$t_0 = 0.664206(1)$ $t_1 = 0.428941(2)$ $t_2 = 1.000000(2)$ $t_3 = 0.000000(3)$ $t_4 = 0.644267(3)$ $t_5 = 0.509683(4)$	$t_0 = 0.163671(2)$ $t_1 = 1.000000(2)$ $t_2 = 0.000000(3)$ $t_3 = 0.825578(3)$	$t_0 = 0.101609(1)$ $t_1 = 0.612460(1)$ $t_2 = 0.296948(2)$ $t_3 = 0.170846(3)$	$t_0 = 0.112691(1)$ $t_1 = 0.550701(1)$ $t_2 = 1.000000(1)$ $t_3 = 0.000000(2)$ $t_4 = 0.540463(2)$ $t_5 = 0.700002(2)$ $t_6 = 0.700002(2)$ $t_7 = 0.798740(2)$
The classified of characteristic points		$t_1 t_2$ are singular points		$t_2 t_3$ are cusp points t_5 is a singular point t_7 is a singular point
The coordinate values of characteristic points	t_0(420, 304) t_1(643, 287) t_2(797, 289) t_3(797, 289) t_4(939, 321) t_5(1106, 314)	t_0(237, 232) t_1(653, 663) t_2(653, 663) t_3(1077, 245)	t_0(335, 33) t_1(250, 46) t_2(285, 251) t_3(554, 370)	t_0(304, 100) t_1: (222, 148) t_2(25, 688) t_3(25, 688) t_4(1086, 499) t_5(1122, 493) t_6(1111, 492) t_7(1070, 484)

dots represents control points of cubic Bézier curves generated by divided cubic B-spline curve, dark lines connected constitute control polygon of Bézier curves. The small solid rectangle represents non-singular inflection points of cubic B-spline curve, the big solid rectangle represents singular points, **+** represents cusp points.

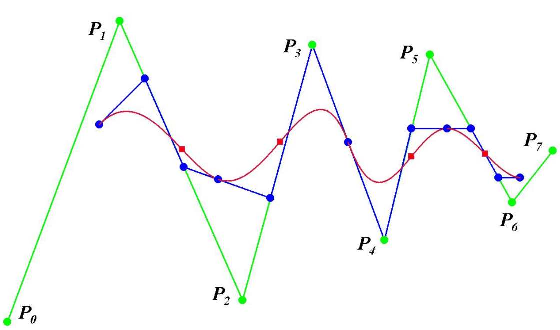

Fig. 6. A non-uniform cubic B-spline curve contains only non-singular inflection points

TABLE I shows the knot vectors and number of divided

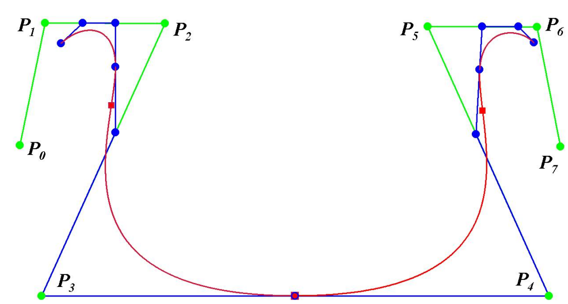

Fig. 7. A cubic B-spline curve contains non-singular inflection points and singular points

cubic Bézier curves corresponding to Fig.6 - Fig.9. TABLE II shows the control points coordinates values of curves from Fig.6 to Fig.9, the classified of shape characteristic points, the corresponding parameter values and coordinates values. The upper-left corner is the coordinate origin point, rightward is the X-axis' positive direction, downward is the Y-axis' positive direction. Where shape characteristic points are non-singular

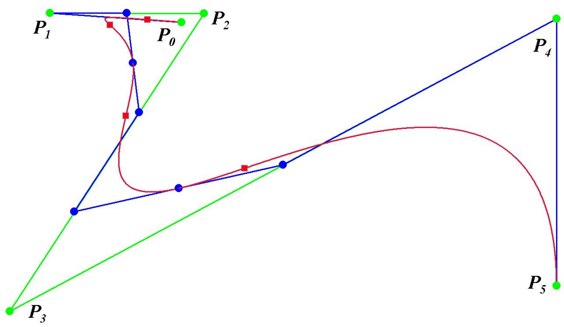

Fig. 8. A uniform cubic B-spline curve with multiple knot vectors

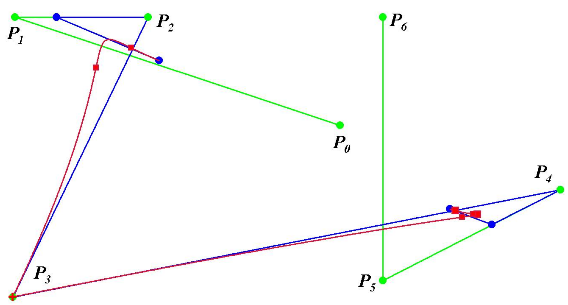

Fig. 9. A B-spline curve contains non-singular infection points,cusp points and singular points

inflection points without special statement. The parameters value of cubic Bézier curves retain only six digits after the decimal point, numbers in brackets indicates shows that the parameters belong to which Bézier curves. The coordinates values of the shape characteristic points are approximative by substituted the corresponding parameters into the curve equation and taken the integer values of them.

V. Conclusion

The formula for calculating the inflection points which represent the shape characteristic of planar cubic Bézier curve were given based on the analysis of curvature expression. Combined the shape characteristic of the control polygons and classified inflection points, we proposed a algorithm which can quickly and correctly obtain shape characteristic points of cubic Bézier curve and B-spline curve. The experimental results verify the correctness of the formula and excellent robustness of the algorithm.

Acknowledgment

This work was supported by Northwest A&F University of China and Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry.

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